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Geometric theory of trigger waves – A dynamical system approach

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We propose a geometric model for wave propagation in excitable media. Our model is based on the Fermat principle and it resembles that of Wiener and Rosenblueth. The model applies to the propagation of excitations, such as chemical and biological wave fronts, grass fire, etc. Starting from the Fermat principle, some consequences of the assumptions are derived analytically. It is proved that the model describes a dynamical system, and that the wave propagates along "ignition lines" (extremals). The theory is applied to the special cases of tube reactor and annular reactor. The asymptotic shape of the wave fronts is derived for these cases: they are straight lines perpendicular to the tube, and involutes of the central obstacle, respectively.

1. Introduction

The first detailed geometrical description of wave propagation in an excitable medium was published by Wiener and Rosenblueth in 1946 [20]. They modelled the propagation of impulses in the nervous system and cardiac muscle. Their postulates were:

(1) The velocity of propagation is constant, it depends neither on the space nor the direction.

(2) The amplitude of the process remains constant and exceeds the threshold.

(3) The points of the medium may assume three states:

   (a) active state (the points in this state constitute the front line),

   (b) refractory state (this state occurs in the rear of the front),

   (c) resting state (in this state the point is excitable).

To model the propagation of excitation satisfying the above assumption, they assigned an *epoch number* to every point to characterize its state. The epoch number u is a function of space and time: $u(x, t)$. The rules for u are:

(i) $u(x, t) = 0$, if x is in the active state at the instant t. Immediately after that u

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increases linearly in time with a constant rate k until u reaches the value 1. Therefore, in the refractory state $0 < u < 1$.

(ii) $u(x, t) = 1$, if x is in the resting state. This value remains unchanged in time unless the point x "is in contact with 0" (that is with an active point). These rules govern the propagation of the front.

We note that the latest rule is well defined in the case of discrete cellular automaton, which is a widely used model for the numerical investigation of the wave propagation in excitable media. In our continuous model the term "contact" is replaced by the topological concept "boundary point".

Wiener and Rosenblueth detailed the consequences of their model for waves rotating in a closed circular ring (one-dimensional problem), and for waves propagating in a convex region of the plane and also in a convex region with convex obstacles (two-dimensional problems). Using the Huygens principle, they stated that "the successive wave fronts are perpendicular to a system of rays which represent the position which may be assumed by stretched cords starting from the stimulated point and passing around all obstacles". That means for the case of one convex obstacle that the front lines are involutes of that obstacle.

One of the most widely studied fields of trigger waves is the chemical wave. In a reacting medium the concentration profiles give the basis of the description of waves. In contrast to the theory of Wiener and Rosenblueth, for chemical waves the front lines are not so straightforwardly defined; we can assign front lines, for example, to a given value or a local maximum value of the concentration of a given chemical component, or to the inflexion point of the concentration profile, and so on. In this context the width of the front can also be studied, see e.g. [14].

In the literature on the geometric theory of chemical waves one can distinguish two main trends. The first one originates in the works of Luther (1906), Kolmogoroff, Petrovsky and Piscounoff (1937), and Fisher (1937) [12,15,16]. These works, belonging to that trend, start from the reaction–diffusion equation, and using singular perturbation theory they derive some qualitative properties of the wave front. A good review of this method is given in [11,19], and in [5], where the eikonal equation is also derived.

In the other trend, which originates in the epoch number concept of Wiener and Rosenblueth, the authors investigate cellular automaton models. The usual models can describe the shape and velocity of the waves [3,4,11], but there are models which simulate the dispersion relation and curvature effect, too [2].

Now we follow another route. We construct a simple mathematical model for the wave propagation. We use strictly defined terminology and assumptions which are different from those in Wiener and Rosenblueth's theory in some respects. Our main goal is to embed the model into the dynamical systems theory. This model is simpler to manage mathematically than the reaction–diffusion equation, but it cannot describe some features of the wave propagation. It is possible to develop a more realistic model especially considering the process of "resurrection" [17]. Some consequences of this simple geometric theory have already been used to explain some experimental facts concerning wave propagation for the Belousov–Zhabotinsky system in modified membranes [9].

Here we consider only two-dimensional regions. Our assumptions are:

(1) Any point of the reactor V may assume three states: active, excitable, and dead. The set of the active points is denoted by A: this is the front line. The set of the excitable points is denoted by E: this is the region in which wave propagation is possible. The remainder part of the reactor V consists of dead points.

In this paper we will not consider the process of "resurrection", therefore the dead points remain dead forever. This restriction does not affect the problems we want to investigate (dynamical system property, extremals, asymptotic wave shape), but with this restriction we abdicate the investigation of spiral waves and the dispersion relation.

(2) The sets A and E depend on time. We assume that any active point continuously activates the excitable points in its "contact" neighbourhood. To be more precise, let us denote the velocity of this activating process by v. To each curve γ, which starts from an active point P and of which all other points are excitable, we assign a time duration

$$\tau = \int_{\gamma} ds/v\,, \tag{1}$$

which the propagation takes to go along the curve. After time τ the point P and the inner points of γ will be dead. The final points Q of γ will be

- dead if there exists another curve γ^* starting from an active point (not necessarily P), going along excitable points and ending at the given point Q, which has a propagation time $\tau^* < \tau$,
- active otherwise.

This assumption is straightforward for propagation of grass fire [18]: the activation at a point is elicited by the first possible impulse, the later impulses have no effect at this point. Also, this assumption is essentially the same as the Fermat principle for the propagation of light.

Hence we are able to determine the time dependence of the sets A and E.

(3) We assume that the integral in (1) exists for all rectifiable curves (that is for curves with finite length). Furthermore, we assume that the velocity v is a function of the space and the direction (for a smooth curve, the direction is determined by the tangent), and v is positive and bounded. In this respect we allow more general cases than Wiener and Rosenblueth do (they restricted themselves to the case of constant velocity). On the other hand, we exclude the cases when the velocity depends on the shape of wave fronts, so we neglect the curvature effect.

In section 2, we establish a simple geometrical model of chemical waves, and prove that this model has the dynamical system properties, which means that the evolution is autonomous. Next we consider the extremals, that is the lines along which the activating process propagates. Finally, we apply our model to two simple kinds of reactors. For a tube reactor we determine the asymptotic shape of the wave front, it will be perpendicular to the tube. For an annular reactor the asymptotic shape of the wave front is an involute of the obstacle. We give an upper bound for the time when the shape exactly takes this involute form.

2. Dynamical system approach of the wave propagation

We deal with wave propagation in the plane. A closed subset V of the plane represents the reactor, that is the region in which the wave can propagate. Let E_0, $A_0 \subset \mathbf{R}^2$ be a disjoint partition of the set V: $V = E_0 \cup A_0$. E_0 denotes the excitable and A_0 the active set at the initial instant. Let us assume that the initial active set A_0 is a closed subset of V.

Let $s(\gamma)$ denote the length of a rectifiable curve γ. In the following we shall consider the curves in V in the arc length parametric form, that is a curve γ will be parametrized by the interval $[0, s(\gamma)]$. Let

$$G := \{\gamma : [0, s(\gamma)] \to V : \gamma \text{ is rectifiable}\}$$

be the set of the rectifiable curves in the set V.

Now we shall define a function τ: $G \to \mathbf{R}_+$, $\tau(\gamma)$ denotes the time that the wave front needs to pass along the curve γ. Let S^1 denote the unit circle in the plane, and let

$$v : V \times S^1 \to \mathbf{R}_+$$

be a continuous velocity function, which gives the velocity of the wave front at a given point and in a given direction. This allows us to define the time $\tau(\gamma)$ as follows:

$$\tau(\gamma) := \int_0^{s(\gamma)} \frac{1}{\nu(\gamma(t), \gamma'(t))} dt . \tag{2}$$

We can use this formula only for piecewise smooth curves. However, in general we do not need the concrete form (2) for the propagation time τ, but three essential properties of it. We shall use a more general definition. Let τ: $G \to \mathbf{R}_+$ be a function which has the following three properties:

(1) If $\gamma \in G$ and $a \in [0, s(\gamma)]$, then

$$\tau(\gamma) = \tau(\gamma|_{[0,a]}) + \tau(\gamma|_{[a,s(\gamma)]}) ,$$

where $\gamma|_{[a,b]}$ denotes the restriction of γ on the interval $[a, b]$.

(2) There exist positive numbers m_1 and m_2, such that for every $\gamma \in G$

$$m_1 s(\gamma) \leqslant \tau(\gamma) \leqslant m_2 s(\gamma).$$

(3) The function τ is continuous, if the topology on G is induced by the Hausdorff metric.

We recall that in the Hausdorff metric [6] the distance between the curves γ and η is

$$d(\gamma, \eta) := \max\{d'(\gamma, \eta), d'(\eta, \gamma)\},$$

where

$$d'(\gamma, \eta) := \sup\{\inf\{|\eta(\beta) - \gamma(\alpha)| : \beta \in [0, s(\eta)]\} : \alpha \in [0, s(\gamma)]\}.$$

Let us introduce the distance between the point x and the set A in the reactor V:

$$d(x, A) := \inf\{\tau(\gamma) : \gamma \in G, \gamma(0) \in A, \gamma(s(\gamma)) = x\}, \quad x \in V, \quad A \subset V.$$

We shall prove that the distance between the points of a curve and a given set changes continuously along the curve.

LEMMA 1

Let $\gamma \in G$ and $A \subset V$. Then the function $\alpha \mapsto d(\gamma(\alpha), A)$ is continuous in the interval $[0, s(\gamma)]$.

Proof

We show that if $|\alpha - \beta| < \epsilon/2m_2$, then

$$|d(\gamma(\alpha), A) - d(\gamma(\beta), A)| < \epsilon. \tag{3}$$

There is a curve $\eta \in G$, such that $\eta(0) \in A$, $\eta(s(\eta)) = \gamma(\alpha)$ and

$$\tau(\eta) < d(\gamma(\alpha), A) + \epsilon/2.$$

Let $\beta \in [0, s(\gamma)]$ such that $|\alpha - \beta| < \epsilon/2\, m_2$. Let $\chi \in G$ be the connection of η and $\gamma|_{[\alpha,\beta]}$, that is the first part of χ is η and the second part of χ is $\gamma|_{[\alpha,\beta]}$, therefore the starting point of χ is in the set A, and the end point of it is $\gamma(\beta)$. Using the properties (1) and (2) of the function τ we get

$$d(\gamma(\beta), A) \leqslant \tau(\chi) = \tau(\eta) + \tau(\gamma|_{[\alpha,\beta]}) < d(\gamma(\alpha), A) + \epsilon,$$

because

$$\tau(\gamma|_{[\alpha,\beta]}) \leqslant m_2\, s(\gamma|_{[\alpha,\beta]}) = m_2\, |\alpha - \beta| < \epsilon/2.$$

Thus $d(\gamma(\beta), A) - d(\gamma(\alpha), A) < \epsilon$, and changing the role of α and β we get

$$d(\gamma(\alpha), A) - d(\gamma(\beta), A) < \epsilon.$$

These two inequalities together give (3), which we had to prove. □

Now we define the excitable set $E(t, E_0, A_0)$, and the active set $A(t, E_0, A_0)$ at time t:

$$E(t, E_0, A_0) := \{x \in V : d(x, A_0) > t\}, \tag{4a}$$

$$A(t, E_0, A_0) := \{x \in V : d(x, A_0) = t\}. \tag{4b}$$

This definition corresponds to our assumptions for the wave propagation process (see assumptions (1) and (2) in the Introduction) and yields a deterministic model: the initial pair (E_0, A_0) uniquely determines the pair (E, A) at any later time. Now we show that the deterministic model given by (4) itself represents a dynamical system. To do this we prove the well-known requirements of dynamical systems, namely the consistency condition for the initial time and the group property [1,13].

THEOREM 1

(i) $E(0, E_0, A_0) = E_0, \quad A(0, E_0, A_0) = A_0.$

(ii) For any $t, s \geqslant 0$

$$E(t + s, E_0, A_0) = E(s, E(t, E_0, A_0), A(t, E_0, A_0)) \text{ and}$$

$$A(t + s, E_0, A_0) = A(s, E(t, E_0, A_0), A(t, E_0, A_0)).$$

Proof

(i) $E(0, E_0, A_0) \subset E_0$ is obvious, and $E_0 \subset E(0, E_0, A_0)$ follows from the closeness of the set A_0.

$A(0, E_0, A_0) \subset A_0$ follows from the closeness of the set A_0, and $A_0 \subset A(0, E_0, A_0)$ is obvious.

(ii) First we prove

$$E(t + s, E_0, A_0) \subset E(s, E(t, E_0, A_0), A(t, E_0, A_0)).$$

Let $x \in E(t + s, E_0, A_0)$, then $d(x, A_0) > t + s \geqslant t$, therefore $x \in E(t, E_0, A_0)$.

We shall show indirectly that $d(x, A(t, E_0, A_0)) > s$. Let us assume the contrary, then for every $\epsilon > 0$ there is a curve $\gamma \in G$ such that $\gamma(0) \in A(t, E_0, A_0))$, $\gamma(s(\gamma)) = x$, and $\tau(\gamma) \leqslant s + \epsilon$. Hence there is another curve $\eta \in G$, which joins A_0 and $\gamma(0)$, that is $\eta(0) \in A_0$, $\eta(s(\eta)) = \gamma(0)$, and $\tau(\eta) \leqslant t + \epsilon$. Let $\chi \in G$ be the connection of η and γ, that is the first part of χ is η and the second is γ. Therefore $\tau(\chi) = \tau(\eta) + \tau(\gamma) \leqslant t + s + \epsilon$; this yields $d(x, A_0) \leqslant t + s$, which is a contradiction.

Now we prove

$$E(s, E(t, E_0, A_0), A(t, E_0, A_0)) \subset E(t+s, E_0, A_0)\,.$$

Let $x \in E(s, E(t, E_0, A_0), A(t, E_0, A_0))$, then $x \in V$ and we have to show that $d(x, A_0) > t+s$. From $x \in E(s, E(t, E_0, A_0), A(t, E_0, A_0))$ follows that there is $\epsilon > 0$ such that for every curve η starting in $A(t, E_0, A_0))$ and ending in the point x, $\tau(\eta) \geqslant s + \epsilon$ holds. We shall show that if $\gamma \in G$, $\gamma(0) \in A_0$, $\gamma(s(\gamma)) = x$, then $\tau(\gamma) \geqslant t + s + \epsilon$, that is $d(x, A_0) > t + s$. Let $h(\alpha := d(\gamma(\alpha), A_0)$, then $h(0) = 0$, $h(s(\gamma)) = d(x, A_0) > t$ and h is continuous; therefore there exists $\beta \in (0, s(\gamma))$ such that $h(\beta) = t$. Hence $\gamma(\beta) \in A(t, E_0, A_0))$ and

$$\tau(\gamma) = \tau(\gamma|_{[0,\beta]}) + \tau(\gamma|_{[\beta, s(\gamma)]}) \geqslant t + s + \epsilon\,.$$

Now let us prove

$$A(t+s, E_0, A_0) \subset A(s, E(t, E_0, A_0), A(t, E_0, A_0))\,.$$

Let $x \in A(t+s, E_0, A_0)$, then $x \in E(t, E_0, A_0)$, and we can prove indirectly that

$$d(x, A(t, E_0, A_0) = s\,.$$

If $d(x, A(t, E_0, A_0) < s$, then $d(x, A_0) < t + s$, which is a contradiction. If $d(x, A(t, E_0, A_0) > s$, then the previous part of this proof shows that $x \in E(t+s, E_0, A_0)$, which is also a contradiction.

Finally we have to show that

$$A(s, E(t, E_0, A_0), A(t, E_0, A_0)) \subset A(t+s, E_0, A_0)\,.$$

Let $x \in A(s, E(t, E_0, A_0), A(t, E_0, A_0))$. Using this previous part of this proof it is easy to prove indirectly that $d(x, A_0) = t + s$. □

3. Extremals (ignition lines)

A more constructive way to determine the process of wave propagation is to define the extremals, that is the curves along which propagation takes places.

According to the pictorial model of grass fire it is straightforward to assume that in the wave model the action may propagate in the excitable media along any curve starting from an active point. The action could reach an excitable point along different curves, may be starting from different active points, but really the action propagates only along exceptional curves, *extremals* determined by the Fermat minimal time principle. In the activation process there is no difference between the active points in activity, the intensity does not play any role. (See Introduction, assumption (2).)

THEOREM 2

For every $x \in V$ there exists $\gamma \in G$, such that $\gamma(0) \in A_0$, $\gamma(s(\gamma)) = x$, and $\tau(\gamma) = d(x, A_0)$. We refer to γ as an extremal.

Proof

From the definition of $d(x, A_0)$ follows that there is a sequence of curves $(\gamma_n) \subset G$, such that $\gamma_n(0) \in A_0$, $\gamma_n(s(\gamma_n)) = x$ and $\tau(\gamma_n)$ tends to $d(x, A_0)$ when n tends to infinity. Let $D := d(x, A_0)$. From property (2) of τ we get that $s(\gamma_n) \leqslant 2D/m_1$ for every n (we can assume that $\tau(\gamma_n) \leqslant 2D$). Let us extend the functions γ_n from the $[0, s(\gamma_n)]$ to the interval $[0, 2D/m_1]$, so that $\gamma_n(\alpha) = x$ for every $\alpha \in [s(\gamma_n), 2D/m_1]$. Therefore we have a sequence of functions $\gamma_n : [0, 2D/m_1] \to V$, which is bounded and equicontinous, because

$$|\gamma_n(\alpha) - \gamma_n(\beta)| \leqslant s(\gamma_n|_{[\alpha,\beta]}) \leqslant |\alpha - \beta| .$$

Hence we can apply the Arzela–Ascoli theorem; therefore there is a uniformly convergent subsequence of (γ_n), which tends to a continuous curve γ. It is easy to prove that the limit of a uniformly convergent sequence of rectifiable curves is also rectifiable if the length of the curves is bounded, that is γ is a rectifiable curve. Since the set A_0 is closed therefore $\gamma(0) \in A_0$. It is obvious that the sequence γ_n tends to γ in the Hausdorff metric too, hence property (3) of τ yields that $\tau(\gamma) = d(x, A_0)$. Thus γ is an extremal. □

This theorem guaranties the existence of the extremals, but does not state anything about uniqueness. Indeed, several points of A_0 can activate the same excitable point at time t. On the other hand, an active point can activate several excitable points at the same time.

4. Tube reactor

Consider a two dimensional semi-infinite rectangular tube reactor with width L (Fig. 1). Assuming constant velocity we prove that the asymptotic shape of the front (that is the shape which the front takes after long time, i.e. far from the initial active set) is a line segment, which is perpendicular to the walls of the tube.

Now the reactor is: $V = \mathbf{R}_+ \times [0, L]$. For the sake of simplicity, we assume that $v(x, y) = 1$ at every $(x, y) \in V$. Therefore $\tau(\gamma)$ is the length of the curve for every rectifiable curve in V, and the extremals are straight lines. We prove the following theorem about the asymptotic shape of the wave front:

THEOREM 3

Let A_0 be a bounded subset in V. Let

$$K := \sup\{x \in \mathbf{R}_+ : \exists y \in [0, L], (x, y) \in A_0\} .$$

Then for every $\epsilon > 0$ there exists $T > 0$ such that if $t > T$, then

$$A(t, E_0, A_0) \subset [K + t - \epsilon, K + t] \times [0, L] .$$

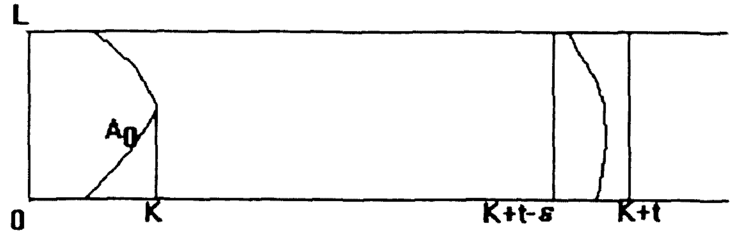

Fig. 1.

This theorem states that the wave front at enough large time is contained in an arbitrary thin (ϵ) perpendicular section of the tube.

Proof

Let $T := \max\{K, (\epsilon^2 + L^2)/2\epsilon\}$ and $t > T$. We show indirectly that if $(x, y) \in A(t, E_0, A_0)$, that is $d(x, y), A_0) = t$, then $x \in [K + t - \epsilon, K + t]$. If $x > K + t$, then $d(x, y), A_0) > t$ for every $y \in [0, L]$, which is a contradiction. If $x < K + t - \epsilon$, then

$$d((x, y), A_0) \leqslant ((t - \epsilon)^2 + L^2)^{1/2}$$

for every $y \in [0, L]$. The definition of T and $t > T$ implies $((t - \epsilon)^2 + L^2)^{1/2} < t$, thus $d((x, y), A_0) < t$, which is a contradiction.

5. Annular reactor

For the case of an annular reactor with constant velocity we show that the wave front will take the form of an involute of the inner curve of the annulus within a finite time [9,20,21]. For the sake of simplicity, the reactor will be the unit disk B centred at the origin without a closed, convex inner part $K \subset B$ (obstacle).

We assume that there exist positive numbers $0 < r_1 < r_2 < 1$, such that $B(r_1) \subset K \subset B(r_2)$. ($B(r)$ denotes the disk with radius r and centred at the origin.) The reactor is the set $V = B \setminus \mathrm{int} K$. Let $v(x, y) = 1$ for simplicity at every $(x, y) \in B \setminus \mathrm{int} K$. Therefore for every rectifiable curve in $B \setminus \mathrm{int} K$, $\tau(\gamma)$ is the length of the curve. The outer boundary of the reactor should contain the disk $B(r_2)$, but might be arbitrary otherwise; for the sake of simplicity we assumed that it is a unit circle.

We need the notion of a supporting line of a convex set [7]:

DEFINITION 1

Let C be a closed convex set on the plane. The line s is a *supporting line* of the set C if:

(i) $C \cap S \neq \oslash$,

(ii) C is in one of the half planes determined by s.

The following elementary lemma will play an important role in our train of thought.

LEMMA 2

Let Γ be a simple closed curve on the plane, and let C be a closed convex set in the bounded domain determined by Γ. Then the circumference of the set C is not greater than the length of the curve Γ.

Now we shall determine the shortest rectifiable curve between two given points P and Q in $B \setminus \operatorname{int} K$, such a curve will be called an *extremal.* Let $s(P, Q)$ denote the segment between two points P and Q.

If the segment $s(P, Q)$ has no common point with the interior of the set K, that is $s(P, Q) \cap \operatorname{int} K = \oslash$, then the extremal curve between P and Q is the segment $s(P, Q)$.

If $s(P, Q) \cap \operatorname{int} K \neq \oslash$, then let e_P and e_Q be those supporting lines which contain P resp. Q, and lie in the right half plane determined by the line PQ (Fig. 2). Similarly, let us denote with f_P and f_Q the supporting lines in the left half plane. Let E_P be the point of $e_P \cap K$ nearest to the point P, and let E_Q be the point of $e_Q \cap K$ nearest to the point Q. The definitions of F_P and F_Q are similar (Fig. 2).

The following statement is a simple consequence of Lemma 2:

LEMMA 3

The extremal between the points P and Q is one of the following curves:

(1) The curve PE_PE_QQ, which consists of the segment $s(P, E_P)$, the border of the set K from the point E_P to the point E_Q and the segment $s(E_Q, Q)$.

(2) The curve PF_PF_QQ, which is similar to the previous one but lies in the other half plane.

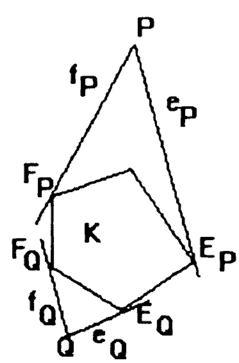

Fig. 2.

If these two curves have the same length, then both of them are extremals.

We recall the definition of the involute and an important property of it [8,10]. The *involutes* of a smooth curve γ are those curves which are orthogonal to the tangent lines of the curve γ. The following proposition is a simple consequence of the definition.

PROPOSITION

Let us consider that involute ρ of the curve γ which contains a point Q of γ (Fig. 3). For every point R of the involute ρ the length of the tangent from R to the curve γ is equal to the arc length of γ between the tangent point and Q.

This proposition gives the following pictorial meaning to the involute: the involute of γ is the locus of the end point of a string which is laid along the curve γ and unwrapped.

We can extend the notion of the involute from the smooth curves to the border of a closed convex set. For this purpose we have to generalize the notion of the tangent to the case of convex sets. A straightforward generalization is the supporting line of a convex set. Let C be a closed convex set, and let P, $Q \in \partial C$ be two points on the border of C. Let L denote the distance of these points on the border (in one of the two possible directions). Let $\gamma : [0, L] \to \partial C$ be the corresponding curve on the border, hence $\gamma(0) = P$ and $\gamma(L) = Q$. The orientation of γ induces an orientation on the supporting lines. We shall divide the supporting lines having a common point with γ in two half lines (Fig. 4). Let s be a supporting line and let S be that point of $s \cap \gamma$ which has the greatest parameter value on γ (S is the last common point). If s has more than one point common with γ, then the orientation on s is yielded trivially. If s and γ has only one common point (S in Fig. 4), then the introduction of the orientation on s can also be carried out uniquely in a natural way. Let us denote the half line corresponding to the induced positive orientation by s^+. We call s^+ the *positive supporting half line*.

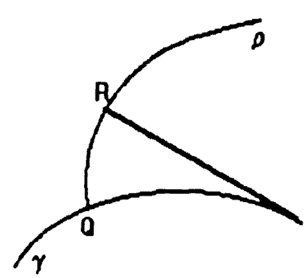

Fig. 3.

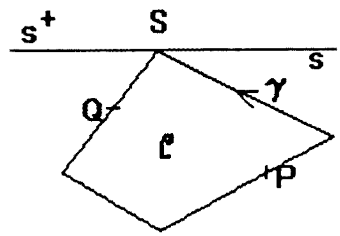

Fig. 4.

DEFINITION 2

The involutes of the curve γ are those curves which are orthogonal to the positive supporting half lines of the convex set C.

REMARK

The rigorous definition of the involute would need the notion of the *positive* tangent even in the case of smooth curves, because the unique construction of the orthogonal trajectories requires that different tangent lines have no common point.

EXAMPLE

Let C be the triangle ABC, and let γ be the union of the segments CB and BA (Fig. 5). The involute of the curve γ containing the point A is a circle with centre at the point B in the upper half plane determined by the line BC, in the lower half plane the involute is a circle with centre at the point C.

Now we show that the property of the involutes formulated in the above proposition for smooth curves is preserved for the present involute concept of border lines of convex sets. Let R be a point outside the convex set C, we shall denote with $s^+(R)$ the positive supporting half line which contains the point R, and we denote by $E(R)$

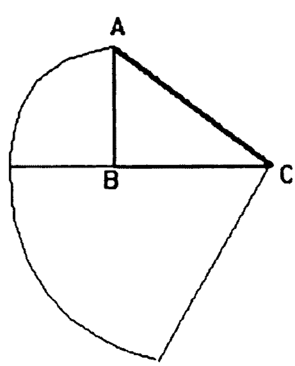

Fig. 5.

the point of $s^+(R) \cap \gamma$ closest to R (Fig. 6). Using Definition 2, the following lemma is obvious.

LEMMA 4

Let γ be the curve on the border of the convex set C as we defined in connection with s^+ (Fig. 6). Let us consider that involute ρ of the curve γ which contains the point Q. For every point R of the involute ρ the length of the segment joining the point R with the point $E(R)$ is equal to the arc length of γ between the point $E(R)$ and Q.

Now we can revert to the problem of annular reactors. Let us recall that K is a closed convex set and the annular reactor is $B \setminus \mathrm{int}\, K$. From Lemma 3 and Lemma 4, one can get the following

COROLLARY

Let $P, Q \in \partial K$ be two points on the border of the set K, and L be the arc length of the border ∂K between the points P and Q (in one direction). Let s be a supporting line containing the point P; let us denote with h that half plane determined by the line s which contains the point Q. Let ρ be that involute of ∂K which contains the point Q. Then for every $R \in \rho \cap h$ the length of the shortest curve in V from the point R to the point P is L.

Before we state our main theorem we have to specify the initial wave front A_0 from which the front will be an involute after finite time.

Let $F \in \partial K$ and $G \in \partial B$ be two points, such that the segment $s(F, G) \subset B \setminus \mathrm{int}\, K$ (Fig. 7). Let t be a supporting line of the set K, which has no common point with the segment $s(F, G)$. Let $T \in t \cap \partial K$ be a point, and let us denote by M and N the intersection points of the line t with the unit circle ∂B. We shall denote by $MTFG$ the closed domain determined by the simple closed curve, which consists of the following four parts: the segment $s(M, T)$; the border of the set K between the points T and F; the segment $s(F, G)$; the arc of the unit circle ∂B between the points G and M. Similarly we shall use the notation $NTFG$ for the corresponding domain.

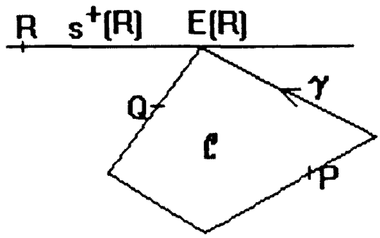

Fig. 6.

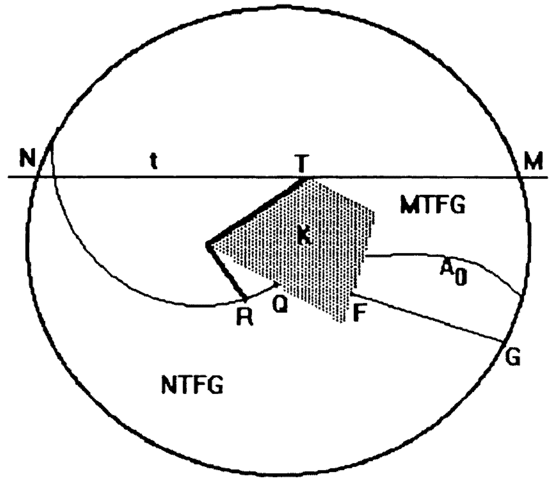

Fig. 7.

Starting from an initial wave front the wave propagates both forward and backward (anticlockwise and clockwise). In order to allow a wave (say anticlockwise) to make a whole round we should "kill" the other (clockwise) one. For this reason we redefine the reactor V excluding the segment $s(F, G)$.

THEOREM 4

Let $A_0 \subset B \setminus \operatorname{int} K$ be a closed set for which there exist the points $F \in \partial K$, $G \in \partial B$ and the supporting line t according to the conditions above, moreover $A_0 \subset MTFG$ and $A_0 \cap s(F, G) = \emptyset$. Let $V := B \setminus \operatorname{int} K \setminus s(F, G)$ and $E_0 := V \setminus A_0$. Then the part of the wave front $A(t, E_0, A_0)$ contained in the domain $NTFG$ is an involute of the border of the set K.

REMARK

The theorem states that a part of the wave front will be an involute in a finite time (when the front reaches the domain $NTFG$). The definition of the set V causes that the front moves only anticlockwise in the annulus, because it cannot cross the segment $s(F, G)$.

Proof

The essential point of the proof is the following: if R is a point in the domain $NTFG$, then the shortest curve from R to A_0 must contain the point T, that is the front in the domain $NTFG$ is the same as the initial front would be only at the point T; in other words, T is the ignition point of the front in the domain $NTFG$.

According to Lemma 3, the shortest curve from R to A_0 passes through one of the supporting lines determined by the point R. But one of them is excluded because the front cannot cross the segment $s(F, G)$; therefore the shortest curve contains the point T.

We recall that $d(P, A_0)$ denotes the length of the shortest curve between a point P and A_0. Let $l := d(T, A_0)$. Hence $d(R, A_0) = l + d(R, T)$, where $d(R, T)$ denotes the length of the shortest curve between the points R and T. Let $L > l$ be a number for which the point $Q \in \partial K$ determined by the equality $d(Q, T) = L - l$ is on the arc ∂K between the points T and F. Then for any point $R \in NTFG$ the equality $d(R, A_0) = L$ is equivalent to $d(R, T) = L - l$, which is equivalent, according to the corollary, to the fact that R is on the involute containing the point Q. This completes the proof. □

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